Structure of Three-Manifolds
– Poincaré and geometrization conjectures

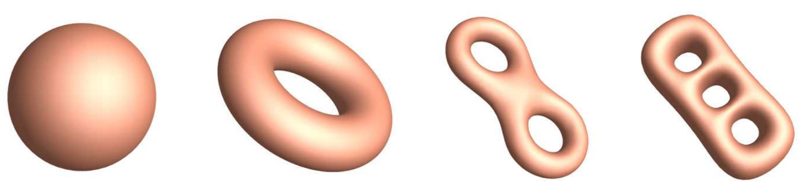
Shing-Tung Yau[1,2]

Ladies and gentlemen, today I am going to tell you the story of how a chapter of mathematics has been closed and a new chapter is beginning.

Let me begin with some elementary observations.

A major purpose of Geometry is to describe and classify geometric structures of interest. We see many such interesting structures in our day-to-day life.

Let us begin with topological structures of a two dimensional surface. These are spaces where locally we have two degrees of freedom. Here are some examples:



genus 0 genus 1 genus 2 genus 3

Genus of a surface is the number of handles of the surface.

An abstract and major way to construct surfaces is by connecting along some deleted disk of each surface.

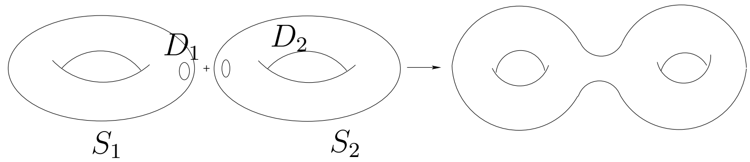
The connected sum of two surfaces S_1 and S_2 is denoted by $S_1 \# S_2$. It is formed by deleting the interior of disks D_i from each S_i and attaching the resulting punctured surfaces $S_i - D_i$ to each other by a one-to-one continuous

[1] This was a talk given at the Morningside Center of Mathematics on June 20, 2006.

[2] All the computer graphics are provided by David Gu, based on the joint paper of David Gu, Yalin Wang and S.-T. Yau.

map $h : \partial D_1 \to \partial D_2$, so that

$$S_1 \# S_2 = (S_1 - D_1) \cup_h (S_2 - D_2).$$



Example:

A genus 8 surface, constructed by connected sum.

The major theorem for the two dimensional surfaces is the following:

Theorem (Classification Theorem for Surfaces). *Any closed, connected orientable surface is exactly one of the following surfaces: a sphere, a torus, or a finite number of connected sum of tori.*

Note that a surface is called orientable if each closed curve on it has a well-defined continuous normal field.

1 Conformal geometry

In order to understand surfaces in a deep manner, Riemann, Poincaré and others proposed to study conformal structure on these two dimensional objects. Such structures allow us to measure angles in the neighborhood of each point on the surface.

For example, if we take a standard atlas of the globe, we have longitude and latitude. They are orthogonal to each other. When we map the atlas, which is a square, onto the globe; distances are badly distorted. For example, the region around the north pole is shown to be a large region on the square. However, the fact that longitude and latitude is orthogonal to each other is preserved under the map. Hence if a ship moves in the ocean, we can use the atlas to determine its direction accurately, but not the distance travelled.

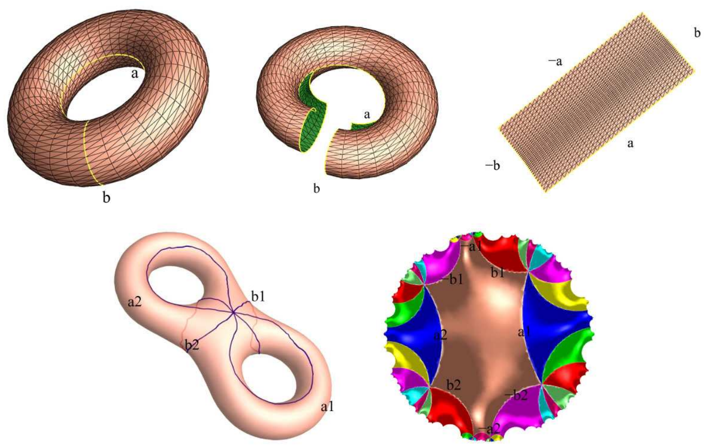
Globe

Poincaré found that at any point, we can draw a longitude (blue curve) and latitude (red curve) on any surface of genus zero in three space. These curves are orthogonal to each other and they converge to two distinct points, on the surface, just like north pole and south pole on the sphere. This theorem of Poincaré also works for arbitrary abstract surface with genus zero.

It is a remarkable theorem that for any two closed surfaces with genus zero, we can always find a one-to-one continuous map mapping longitude and latitude of one surface to the corresponding longitude and latitude of the other surface. This map preserve angles defined by the charts. In such a situation, we say that these two surfaces are conformal to each other. And there is only one conformal structure for a surface with genus zero.

For genus equal to one, the surface looks like a donut, and we can draw longitude and latitude with no north or south poles. However, there can be distinct surfaces with genus one that are not conformal to each other. In fact, there are two parameters of conformal structures on a genus one surface. For genus g greater than one, one can still draw longitude and latitude (the definition of such curves needs to be made precise). But they have many poles, the number of which depends on the genus. The number of parameters of conformal structures over a surface with genus g is $6g-6$.

In order to find a global atlas of the surface, we can cut along some special curves of a surface and then spread the surface on the plane or the disk. In this procedure, the longitude and the latitude will be preserved.



A fundamental theorem for surfaces with metric structure is the following theorem.

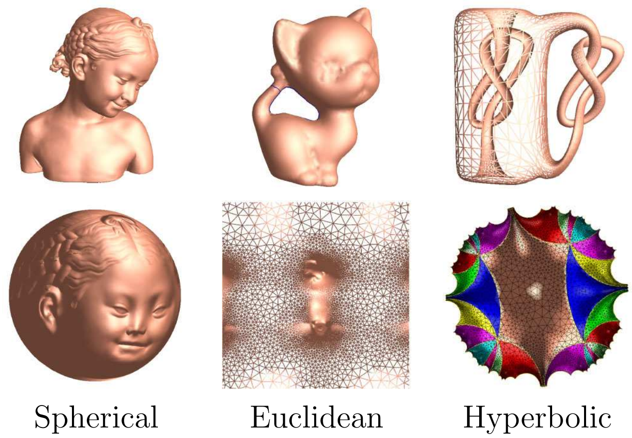
Theorem (Poincaré's Uniformization Theorem). *Any closed two-dimensional space is conformal to another space with constant Gauss curvature.*

- *If curvature* >0*, the surface has genus* $=0$*;*
- *If curvature* $=0$*, the surface has genus* $=1$*;*
- *If curvature* <0*, the surface has genus* >1*.*

The generalization of this theorem plays a very important role in the field of geometric analysis. In particular, it motivates the works of Thurston and Hamilton. This will be discussed later in this talk.

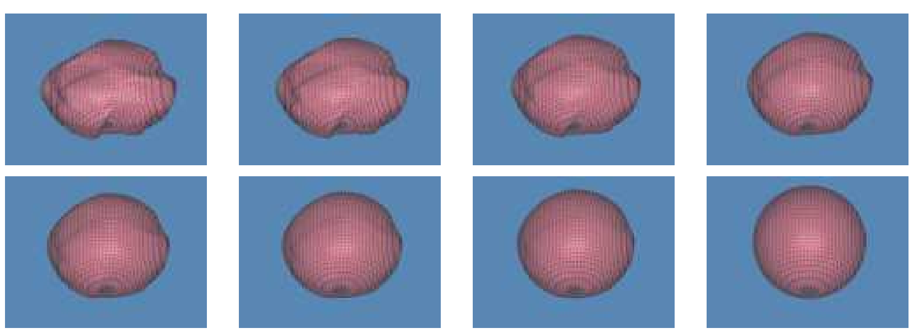
2 Hamilton's equation on Surfaces

Poincaré's theorem can also be proved by the equation of Hamilton. We can deform any metric on a surface by the negative of its curvature. After



normalization, the final state of such deformation will be a metric with constant curvature. This is a method created by Hamilton to deform metrics on spaces of arbitrary dimensions. In higher dimension, the typical final state of spaces for the Hamilton equation is a space that satisfies Einstein's equation.

As a consequence of the works by Richard Hamilton and B. Chow, one knows that in two dimension, the deformation encounters no obstruction and will always converge to one with constant curvature. This theorem was used by David Gu, Yalin Wang, and myself for computer graphics. The following sequence of pictures is obtained by numerical simulation of the Ricci flow in two dimension.



3 Three-Manifolds

So far, we have focused on spaces where there are only two degrees of freedom. Instead of being a flat bug moving with two degrees of freedom on a surface, we experience three degrees of freedom in space. While it seems that our three dimensional space is flat, there are many natural three dimensional spaces, which are not flat.

Important natural example of higher dimensional spaces are phase spaces in mechanics.

In the early twentieth century, Poincaré studied the topology of phase space of dynamics of particles. The phase space consists of $(x; v)$, the position and the velocity of the particles. For example if a particle is moving freely with unit speed on a two dimensional surface Σ, there are three degrees of freedom in the phase space of the particle. This gives rise to a three dimensional space M.

Such a phase space is a good example for the concept of fiber bundle.

If we associate to each point $(x; v)$ in M the point $x \in \Sigma$, we have a map from M onto Σ. When we fix the point x, v can be any vector with unit length. The totality of v forms a circle. Therefore, M is a fiber bundle over Σ with fiber equal to a circle.

4 The Poincaré Conjecture

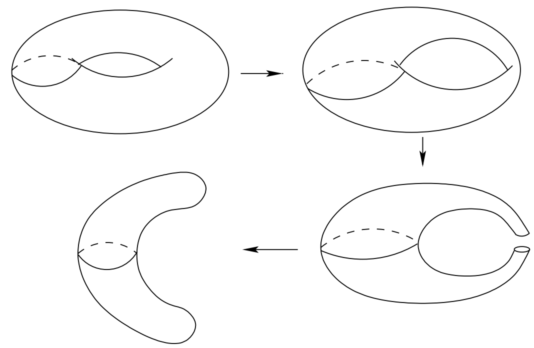
The subject of higher dimensional topology started with Poincaré's question:

Is a closed three dimensional space topologically a sphere if every closed curve in this space can be shrunk continuously to a point?

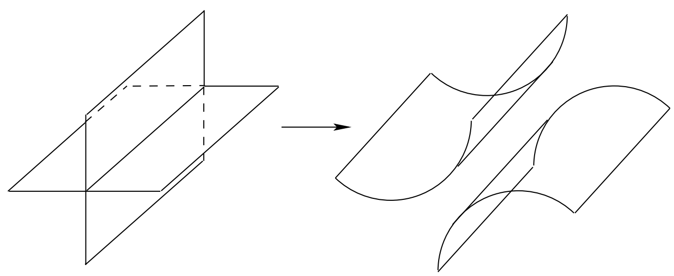
This is not only a famous difficult problem, but also the central problem for three dimensional topology. Its understanding leads to the full structure theorem for three dimensional spaces. I shall describe its development chronologically.

5 Topological Surgery

Topologists have been working on this problem for over a century. The major tool is application of cut and paste, or surgery, to simplify the topology of a space:



Two major ingredients were invented. One is called Dehn's lemma which provides a tool to simplify any surface which cross itself to one which does not.



Theorem (Dehn's lemma) *If there exists a map of a disk into a three dimensional space, which does not cross itself on the boundary of the disk, then there exists another map of the disc into the space which does not cross itself and is identical to the original map on the boundary of the disc.*

This is a very subtle theorem, as it took almost fifty years until Papakyriakopoulos came up with a correct proof after its discovery.

The second tool is the construction of incompressible surfaces introduced by Haken. It was used to cut three manifolds into pieces. Walhausen proved important theorems by this procedure. (Incompressible surfaces are embedded surfaces which have the property whereby if a loop cannot be shrunk to a point on the surface, then it cannot be shrunk to a point in the three dimensional space, either.)

6 Special Surfaces

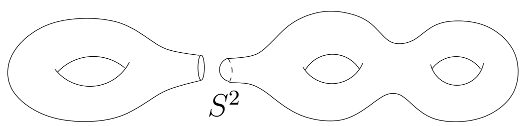
There are several important one dimensional and two dimensional spaces that play important roles in understanding three dimensional spaces.

1. Circle

Seifert constructed many three dimensional spaces that can be described as continuous family of circles. The above mentioned phase space is an example of a **Seifert space**.

2. Two dimensional spheres

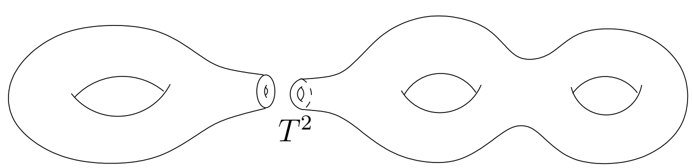
We can build three dimensional spaces by removing balls from two distinguished ones and gluing them along the boundary spheres. Conversely



Kneser and Milnor proved that each three dimensional space can be uniquely decomposed into irreducible components along spheres. (A space is called irreducible if each embedded sphere is the boundary of a three dimensional ball in this space.)

3. Torus

A theorem of Jaco-Shalen, Johannson says that one can go one step further by cutting a space along tori.



7 Structure of Three Dimensional Spaces

A very important breakthrough was made in the late 1970s by W. Thurston. He make the following conjecture.

Geometrization Conjecture (Thurston): The structure of three dimensional spaces is built on the following **atomic** spaces:

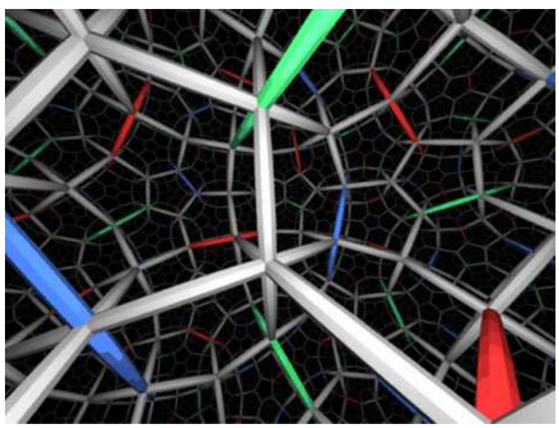
(1) The Poincaré conjecture: three dimensional space where every closed loop can be shrunk to a point; this space is conjectured to be the three-sphere.

(2) The space-form problem: spaces obtained by identifying points on the three-sphere. The identification is dictated by a finite group of linear isometries which is similar to the symmetries of crystals.

(3) Seifert spaces mentioned above and their quotients similar to (2).

(4) Hyperbolic spaces according to the conjecture of Thurston: three-space whose boundaries may consist of tori such that every two-sphere in the space is the boundary of a ball in the space and each incompressible torus can be deformed to a boundary component; it was conjectured to support a canonical metric with constant negative curvature and it is obtained by identifying points on the hyperbolic ball. The identification is dictated by a group of symmetries of the ball similar to the symmetries of crystals.

An example of a space obtained by identifying points on the three dimensional hyperbolic space



Hyperbolic Space Tiled with Dodecahedra,
by Charlie Gunn (Geometry Center).
from the book "Three-dimensional geometry and topology"
by Thurston, Princeton University press

Thurston's conjecture effectively reduced the classification of three dimensional spaces to group theory, where many tools were available. He and his followers proved the conjecture when the three space is sufficiently large in the sense of Haken and Walhausen. (A space is said to be sufficiently large if there is a nontrivial incompressible surface embedded inside the space. Haken and Walhausen proved substantial theorem for this class of manifolds.) This theorem of Thurston covers a large class of three dimensional hyperbolic manifolds.

However, as nontrivial incompressible surface is difficult to find on a general space, the argument of Thurston is difficult to use to prove the Poincaré conjecture.

8 Geometric Analysis

On the other hand, starting in the seventies, a group of geometers applied nonlinear partial differential equations to build geometric structures over a space. Yamabe considered the equation to conformally deform metrics to metrics with constant scalar curvature. However, in three dimension, metrics with negative scalar curvature cannot detect the topology of spaces.

A noted advance was the construction of Kähler-Einstein metrics on Kähler manifolds in 1976. In fact, I used such a metric to prove the complex version of the Poincaré conjecture. It is called the Severi conjecture in complex geometry. It says that every complex surface that can be deformed to the complex projective plane is itself the complex projective plane.

The subject of combining ideas from geometry and analysis to understand geometry and topology is called **geometric analysis**. While the subject can be traced back to 1950s, it has been studied much more extensively in the last thirty years.

Geometric analysis is built on two pillars: nonlinear analysis and geometry. Both of them became mature in the seventies based on the efforts of many mathematicians. (See my survey paper "Perspectives on geometric analysis" in *Survey in Differential Geometry*, Vol. X. 2006.)

9 Einstein metrics

I shall now describe how ideas of geometric analysis are used to solve the Poincaré conjecture and the geometrization conjecture of Thurston.

In the case of a three dimensional space, we need to construct an Einstein metric, a metric inspired by the Einstein equation of gravity. Starting from an arbitrary metric on three space, we would like to find a method to deform it to the one that satisfies Einstein equation. Such a deformation has to depend on the curvature of the metric.

Einstein's theory of relativity tells us that under the influence of gravity, space-time must have curvature. Space moves dynamically. The global topology of space changes according to the distribution of curvature (gravity). Conversely, understanding of global topology is extremely important and it provides constraints on the distribution of gravity. In fact, the topology of space may be considered as a source term for gravity.

From now on, we shall assume that our three dimensional space is compact and has no boundary (i.e., closed).

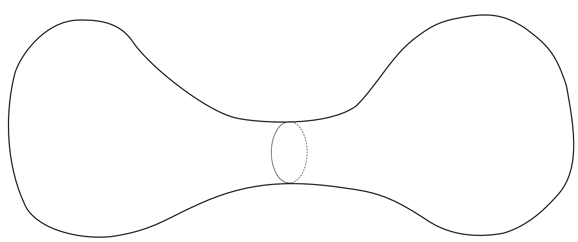
In a three dimensional space, curvature of a space can be different when measured from different directions. Such a measurement is dictated by a quantity R_{ij}, called the **Ricci tensor**. In general relativity, this gives rise to the matter tensor of space.

An important quantity that is independent of directions is the **scalar curvature** R. It is the trace of R_{ij} and can be considered as a way to measure the expansion or shrinking of the volume of geodesic balls:

$$\text{Volume}(B(p,r)) \sim \frac{4\pi}{3}(r^3 - \frac{1}{30}R(p)r^5),$$

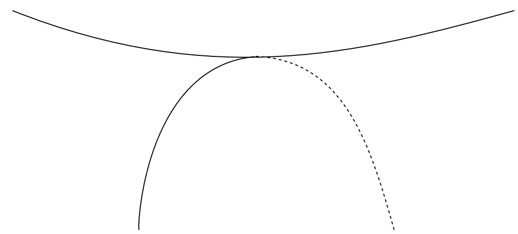
where $B(p,r)$ is the ball of radius r centered at a point p, and $R(p)$ is the scalar curvature at p.

This can be illustrated by a dumbbell surface where, near the neck, curvature is negative and where, on the two ends which are convex, curvature is positive.

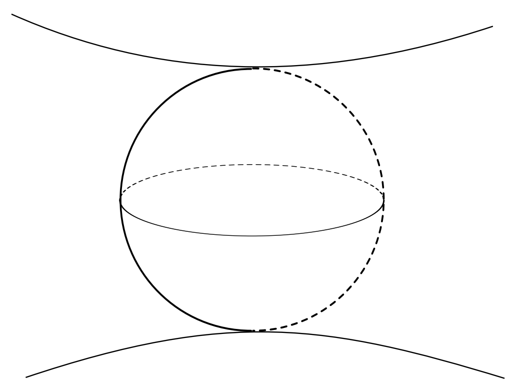


Two-dimensional dumbbell surface

Two-dimensional surfaces with negative curvature look like saddles. Hence a two dimensional neck has negatives curvature. However, in three dimension, the slice of a neck can be a two dimensional sphere with very large



positive curvature. Since scalar curvature is the sum of curvatures in all direction, the scalar curvature at the three dimensional neck can be positive. This is an important difference between a two-dimensional neck and a three-dimensional neck.



Three dimensional neck.

10 The dynamics of Einstein equation

In general relativity, matter density consists of scalar curvature plus the momentum density of space. The Dynamics of Einstein equation drives space to form black holes which splits space into two parts: the part where scalar curvature is positive and the other part, where the space may have a black hole singularity and is enclosed by the apparent horizon of the black hole, the topology tends to support metrics with negative curvature.

There are two quantities in gravity that dictate the dynamics of space: **metric** and **momentum**. Momentum is difficult to control. Hence at this time, it is rather difficult to use the Einstein equation of general relativity to study the topology of spaces.

11 Hamilton's Equation

In 1979, Hamilton developed a new equation to study the dynamics of space metric. The Hamilton equation is given by

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}.$$

Instead of driving space metric by gravity, he drives it by its Ricci curvature which is analogous to the heat diffusion. Hamilton's equation therefore can be considered as a nonlinear heat equation. Heat flows have a regularizing effect because they disperse irregularity in a smooth manner.

Hamilton's equation was also considered by physicists. (It first appeared in Friedan's thesis.) However, this point of view was completely different. Physicists considered it as beta function for deformations of the sigma model to conformal field theory.

12 Singularity

Despite the fact that Hamilton's equation tend to smooth out metric structure, global topology and nonlinear terms in the equation coming from curvature drive the space metric to points where the space topology collapses. We call such points **singularity of space**.

In 1982, Hamilton published his first paper on the equation. Starting with a space with positive Ricci curvature, he proved that under his equation, space, after dilating to keep constant volume, never encounters any singularity and settles down to a space where curvature is constant in every direction.

Such a space must be either a 3-sphere or a space obtained by identifying the sphere by some finite group of isometries.

After seeing the theorem of Hamilton, I was convinced that Hamilton's equation is the right equation to carry out the geometrization program. (This paper of Hamilton is immediately followed by the paper of Huisken on deformation of convex surfaces by mean curvature. The equation of mean curvature flow has been a good model for understanding Hamilton's equation.)

We propose to deform any metric on a three dimensional space which shall break up the space eventually. It should lead to the topological decomposition according to Kneser, Milnor, Jacob-Shalen and Johannson. The asymptotic state of Hamilton's equation is expected to be broken up into pieces which will either collapse or produce metrics which satisfy the Einstein equation.

In three dimensional spaces, Einstein metrics are metrics with constant curvature. However, along the way, the deformation will encounter singularities. The major question is how to find a way to describe all possible singularities. We shall describe these spectacular developments.

13 Hamilton's Program

Hamilton's idea is to perform surgery to cut off the singularities and continue his flow after the surgery. If the flow develops singularities again, one repeats the process of performing surgery and continuing the flow.

If one can prove there are only a finite number of surgeries in any finite time interval, and if the long-time behavior of solutions of the Hamilton's flow with surgery is well understood, then one would be able to recognize the topological structure of the initial manifold. Thus Hamilton's program, when carried out successfully, will lead to a proof of the Poincaré conjecture and Thurston's geometrization conjecture.

The importance and originality of Hamilton's contribution can hardly be exaggerated. In fact, Perelman said:

"The implementation of Hamilton's program would imply the geometrization conjecture for closed three-manifolds."

"In this paper we carry out some details of Hamilton's program".

We shall now describe the chronological development of Hamilton's program. There were several stages:

I. A Priori Estimates

In the early 1990s, Hamilton systematically developed methods to understand the structure of singularities. Based on my suggestion, he proved the fundamental estimate (the Li-Yau-Hamilton estimate) for his flow when curvature is nonnegative. The estimate provides a priori control of the behavior of the flow.

An a prior estimate is the key to proving any existence theorem for nonlinear partial differential equations. An intuitive example can be explained as follows: when a missile engineer designs trajectory of a missile, he needs to know what is the most likely position and velocity of the missile after ten seconds of its launch. Yet a change in the wind will cause reality to differ from his estimate. But as long as the estimate is within a range of accuracy, he will know how to design the missile. How to estimate this range of accuracy is called a prior estimate.

The Li-Yau-Hamilton Estimate

In proving existence of a nonlinear differential equation, we need to find an a priori estimate for some quantity which governs the equation. In the case of Hamilton's equation, the important quantity is the scalar curvature R. An absolute bound on the curvature gives control over the nonsingularity of the space. On the other hand, the relative strength of the scalar curvature holds the key to understand the singularity of the flow. This is provided by the Li-Yau-Hamilton estimate:

For any one-form V_a we have

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2\nabla_a R \cdot V_a + 2R_{ab}V_aV_b \geq 0.$$

In particular, $tR(x,t)$ is pointwise nondecreasing in time.

In the process of applying such an estimate to study the structure of singularities, Hamilton discovered (also independently by Ivey) a curvature pinching estimate for his equation on three-dimensional spaces. It allows him to conclude that a neighborhood of the singularity looks like space with nonnegative curvature. For such a neighborhood, the Li-Yau-Hamilton estimate can be applied.

Then, under an additional non-collapsing condition, Hamilton described the structure of all possible singularities. However, he was not able to show that all these possibilities actually occur. Of particular concern to him was a singularity which he called the cigar.

II. Hamilton's works on Geometrization

In 1995, Hamilton developed the procedure of geometric surgery using a foliation by surfaces of constant mean curvature, to study the topology of four-manifolds of positive isotropic curvature.

In 1996, he went ahead to analyze the global structure of the space time structure of his flow under suitable regularity assumptions (he called them nonsingular solutions). In particular, he showed how three-dimensional spaces admitting a nonsingular solution of his equation can be broken into pieces according to the geometrization conjecture.

These spectacular works are based on deep analysis of geometry and nonlinear differential equations. Hamilton's two papers provided convincing evidence that the geometrization program could be carried out using his approach.

Main Ingredients of these works of Hamilton

In this deep analysis he needed several important ingredients:

(1) a compactness theorem on the convergence of metrics developed by him, based on the injectivity radius estimate proved by Cheng-Li-Yau in 1981. (The injectivity radius at a point is the radius of the largest ball centered at that point that the ball would not collapse topologically.)

(2) a beautiful quantitative generalization of Mostow's rigidity theorem which says that there is at most one metric with constant negative curvature on a three-dimensional space with finite volume. This rigidity theorem of Mostow is not true for two dimensional surfaces.

(3) In the process of breaking up the space along the tori, he needs to prove that the tori are incompressible. The ingredients of his proof depend on the theory of minimal surfaces as was developed by Meeks-Yau and Schoen-Yau.

At this stage, it seems clear to me that Hamilton's program for the Poincaré and geometrization conjectures could be carried out. The major remaining obstacle was to obtain certain injectivity radius control, in terms of local curvature bound, in order to understand the structure of the singularity and the process of surgery to remove the singularity. Hamilton and I worked together on removing this obstacle for some time.

III. Perelman's Breakthrough

In November of 2002, Perelman put out a preprint, "The entropy formula for Hamilton's equation and its geometric applications", wherein major ideas were introduced to implement Hamilton's program.

Parallel to what Li-Yau did in 1986, Perelman introduced a space-time distance function obtained by path integral and used it to verify the noncollapsing condition in general. In particular, he demonstrated that cigar type singularity does not exist in Hamilton's equation.

His distance function can be described as follows.

Let σ be any space-time path joining p to q, we define the action to be

$$\int_0^\tau \sqrt{s}(R + |\dot{\sigma}(s)|^2)ds.$$

By minimizing among all such paths joining p to q, we obtain $L(q, \tau)$.

Then Perelman defined his reduced volume to be

$$\int (4\pi\tau)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sqrt{\tau}}L(q,\tau)\right\}$$

and observed that under the Hamilton's equation it is nonincreasing in τ. In this proof Perelman used the idea in the second part of Li-Yau's paper in 1986. As recognized by Perelman: "in Li-Yau, where they use 'length', associated to a linear parabolic equation, is pretty much the same as in our case."

Rescaling Argument

Furthermore, Perelman developed an important refined rescaling argument to complete the classification of Hamilton on the structure of singularities of Hamilton's equation and obtained a uniform and global version of the structure theorem of singularities.

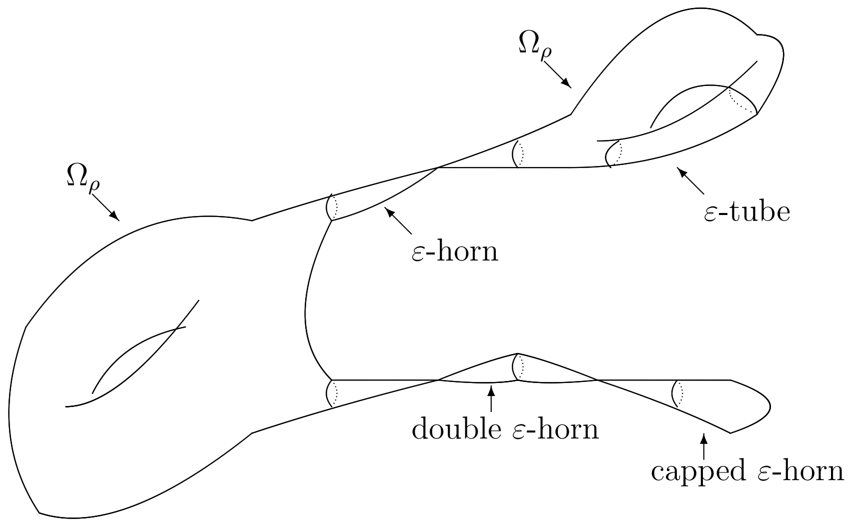
Hamilton's Geometric Surgery

Now we need to find a way to perform geometric surgery. In 1995, Hamilton had already initiated a surgery procedure for his equation on four-dimensional spaces and presented a concrete method for performing such surgery.

One can see that Hamilton's geometric surgery method also works for Hamilton's equation on three-dimensional spaces. However, in order for surgeries to be done successfully, a more refined technique is needed.

Discreteness of Surgery Times

The challenge is to prove that there are only a finite number of surgeries on each finite time interval. The problem is that, when one performs the surgeries with a given accuracy at each surgery time, it is possible that the error may add up to so fast that they force the surgery times to accumulate.



The Structure of Singularity

Rescaling Arguments

In March of 2003, Perelman put out another preprint, titled "Ricci flow with surgery on three manifolds", where he designed an improved version of Hamilton's geometric surgery procedure so that, as time goes on, successive surgeries are performed with increasing accuracy.

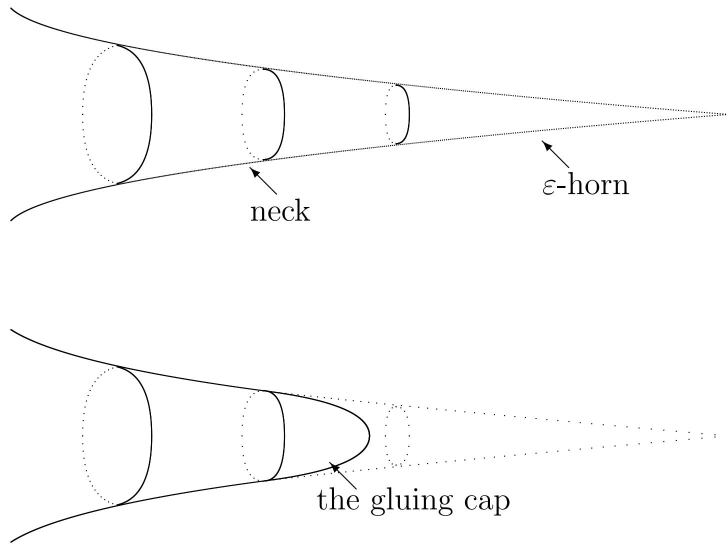
Perelman introduced a rescaling argument to prevent the surgery time from accumulating.

When using the rescaling argument for surgically modified solutions of Hamilton's equation, one encounters the difficulty of applying Hamilton's compactness theorem, which works only for smooth solutions.

The idea of overcoming this difficulty consists of two parts:

1. (Perelman): choose the cutoff radius in the neck-like regions small enough to push the surgical regions far away in space.

2. (Cao-Zhu): establish results for the surgically modified solutions so that Hamilton's compactness theorem is still applicable. To do so, they need a deep understanding of the prolongation of the surgical regions, which in turn relies on the uniqueness theorem of Chen-Zhu for solutions of Hamilton's equation on noncompact manifolds.



geometric surgery

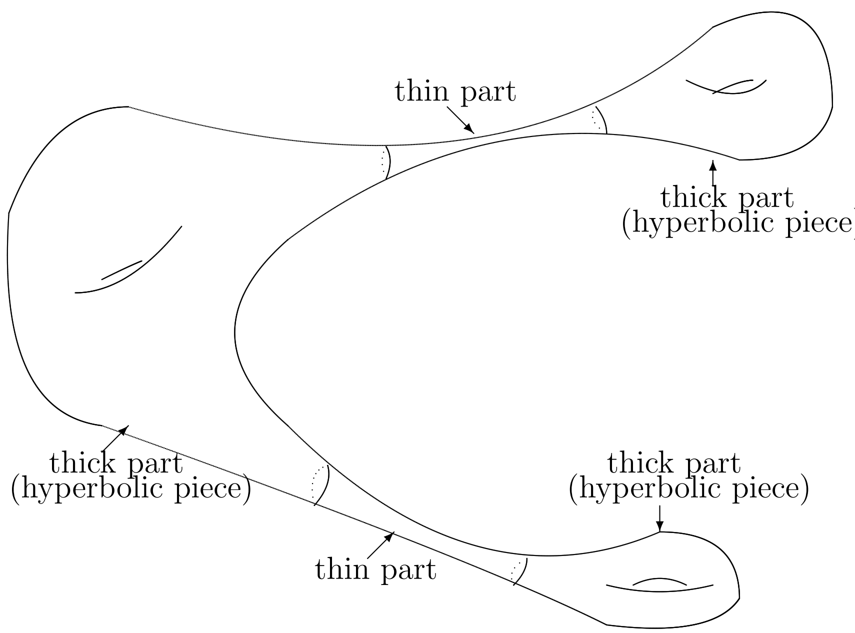
Conclusion of the proof of the Poincaré Conjecture

One can now prove Poincaré conjecture for simply connected three dimensional space, by combining the discreteness of surgeries with finite time extinction result of Colding-Minicozzi (2005).

IV. Proof of the geometrization conjecture: Thick-thin Decomposition

To approach the structure theorem for general spaces, one still needs to analyze the long-time behavior of surgically modified solutions to Hamilton's equation. As mentioned in II, Hamilton studied the long time behavior of his equation for a special class of (smooth) solutions – nonsingular solutions.

In 1996, Hamilton proved that any three-dimensional nonsingular solution admits of a thick-thin decomposition where the thick part consists of a finite number of hyperbolic pieces and the thin part collapses. Moreover, by adapting Schoen-Yau's minimal surface arguments, Hamilton showed that the boundary of hyperbolic pieces are incompressible tori. Consequently, any nonsingular solution is geometrizable.



Thick-thin decomposition

Even though the nonsingularity assumption seems restrictive, the ideas and arguments of Hamilton are used in an essential way by Perelman to analyze the long-time behavior for general surgical solutions. In particular, he also studied the thick-thin decomposition.

For the thick part, based on the Li-Yau-Hamilton estimate, Perelman established a crucial elliptic type estimate, which allowed him to conclude that the thick part consists of hyperbolic pieces. For the thin part, since he could only obtain a lower bound on the sectional curvature, he needs a new collapsing result. Assuming this new collapsing result, Perelman claimed that the solutions to Hamilton's equation with surgery have the same long-time behavior as nonsingular solutions in Hamilton's work, a conclusion which would imply the validity of Thurston's geometrization conjecture.

Although the proof of this new collapsing result was promised by Perelman, it still has yet to appear. (Shioya-Yamaguchi has published a proof of the collapsing result in the special case when the space is closed.) Nonetheless, based on the previous results, Cao-Zhu gave a complete proof of Thurston's geometrization conjecture.

Conclusion

The success of Hamilton's program is the culmination of efforts by geometric analysts in the past thirty years. It should be considered as the crowning achievement of the subject of geometric analysis, a subject that is capable of proving hard and difficult topological theorems by geometry and analysis solely.

Hamilton's equation is a complicated nonlinear system of partial differential equations. This is the first time that mathematicians have been able to understand the structure of singularity and development of such a complicated system.

Similar systems appear throughout the natural world. The methods developed in the study of Hamilton equation should shed light on many natural systems such as the Navier-Stokes equation and the Einstein equation.

In addition, the numerical implementation of the Hamilton flow should be useful in computer graphics, as was demonstrated by Gu-Wang-Yau for two dimensional figures.

Impact on the future of geometry

Poincaré:

"Thought is only a flash in the middle of a long night, but the flash that means everything."

The Flash of Poincaré in 1904 has illuminated a major portion of the topological developments in the last century.

Poincaré also initiated development of the theory of Riemann surfaces. It has been one of the major pillars of all mathematics development in the twentieth century. I believe that the full understanding of the three dimensional manifolds will play a similar role in the twenty-first century.

Remark

In Perelman's work, many key ideas of the proofs are sketched or outlined, but complete details of the proofs are often missing. The recent paper of

Cao-Zhu, submitted to The Asian Journal of Mathematics in 2005, gives the first complete and detailed account of the proof of the Poincaré conjecture and the geometrization conjecture. They substituted several arguments of Perelman with new approaches based on their own studies. The materials were presented by Zhu in a Harvard seminar from September 2005 to March 2006, where faculties and postdoctoral fellows of Harvard University and MIT attended regularly. Some of the key arguments, that has been important for the completion of the Poincare conjecture, has already appeared in the paper of Chen-Zhu [2].

In the last three years, many mathematicians have attempted to see whether the ideas of Hamilton and Perelman can hold together. Kleiner and Lott (in 2004) posted on their web page some notes on several parts of Perelman's work. However, these notes were far from complete. After the work of Cao-Zhu was accepted and announced by the journal in April, 2006 (it was distributed on June 1, 2006). On May 24, 2006, Kleiner and Lott put up another, more complete, version of their notes. Their approach is different from Cao-Zhu's. It will take some time to understand their notes which seem to be sketchy at several important points. Most recently, a manuscript of Morgan-Tian appeared in the web. In a letter to the author, Jim Carlson of the Clay institute stated that the first version of this manuscript was submitted to the Clay institute on May 19, 2006, and the revised version was submitted on July 23, 2006.

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